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STOCHASTIC PROCESSES AND SOME APPLICATIONS

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Abstract

In this paper, we discuss about some applications to the birth and death process in biology and in the theory of waiting. First, we introduce some theoretical notions about the stochastic processes and we define the Markov chain and the Poisson process. Afterwards, we consider two processes of birth: "The Poissonian tide of particles" and " The Yule-Furry process", that have multiple applications, the theory of waiting being one of them. Finally, by using the theoretical concept presented above, we solve two examples: the first is a problem from veterinary medicine, and the second one is an example of the model M/M/1 from agriculture, which means if the entrance, the time of serving and the number of stations are given, we'll calculate the number of the biological units in system, the medium number of the applicants in line and the medium time of waiting.

Key words: *birth and death process, theory of waiting, the Poisson distribution*

INTRODUCTION

Random variables which depend on one or more parameters appear in many applications of probability. The theory of stochastic processes studies the families of random variables defined on the same probability space. Areas such as biology, science, technology and socio-economic are just some applicability domains of this theory. When we consider the reliability of a system (laboratory instrument, agricultural machine, computer, electronic device, etc.) we study the variations of its characteristics over time. Another example (which is just one of the topics of this paper) is about a problem in veterinary medicine, in which we want to determine the number of statistic units of a population from a particular habitat. We can determine the number of particles, bodies, etc., existing in a given space at a time "*t*". A second example solved through the stochastic process theory is an application of serving systems in agriculture.

If we know in terms of probabilities the inputs, outputs and mechanism serving, we study the estimation of the number of statistical units located in the service station at a certain time, the probability of existence of a number of applications in the system, the average waiting time and the average number of busy stations. For discussing and solving the study of the problems proposed in the first stage, it is necessary to define the concepts and basic properties of the theory of stochastic processes and afterwards to determine the solutions, by applying the concepts given above.

MATERIALS AND METHODS

We consider $(\Omega, \mathcal{K}, P)$ a probability space and T a lot.

We define in accordance [1] the notions such as: stochastic process, Markov chains and Poisson process.

Definition 1.

It's called stochastic process (random process or random function), a family of random variables parameterized, $\{X_t\}_{t\in T}$, $X_t : \Omega \to R^n$, $(\forall) t \in T$; T is the *parameter's space* and $S \subset R^n$ the lot of the values of random variables X_t, it's called the *space of state*.

Observations:

-If T=**Z** or T=**N**, then the stochastic process depends on a discreet parameter and it is called chain.

-If T=**R**, $T=[0,\infty)$ or $T=[a,b]$, then the stochastic process depends on a continuous parameter and it is called process with continuous time.

An important criterion for classifying the stochastic processes is determined by the connection between the random variables X_t, when *t* covers through the parameter space. A class of stochastic processes for which the future is independent of the past, as soon as the present is known, is the Markov chains.

Definition 2

The family of random variables $\{X_n\}_{n\in N}$ is called the Markov chain if it verifies the Markov´s relation, i.e. $(\forall)n\geq 2, 0\leq t_1<...<t_n$ and $(\forall)i_1,...,i_n\in S$,

$$P\left[X_{t_n}=i_n \mid X_{t_{n-1}}=i_{n-1},...,X_{t_1}=i_1\right]=P\left[X_{t_n}=i_n \mid X_{t_{n-1}}=i_{n-1}\right] \quad (1)$$

provided that the left member of the previous relationship is defined.

A) The Poisson process is considered the most simple of discontinuous Markov processes and has a special place in the theory of probability, with many applications in biology, physics, engineering, telecommunications and transportation. Such a process is characterized by the following conditions:

a) For the process X_t, the probability to have a change in the time interval $(t,t+\Delta t)$ is $\lambda\Delta t+0(\Delta t)$, where λ is a given positive constant which represents the average number of events produced per unit of time with $\lim\limits_{\Delta t\to 0}\frac{0(\Delta t)}{\Delta t}=0$,

b) The probability to occur more than a change of the process in the time interval $(t,t+\Delta t)$ is $0(\Delta t)$,

c) The probability to occur no change in the time interval $(t,t+\Delta t)$ is $1-\lambda\Delta t-0(\Delta t)$.

These probabilities are independent of the state of the system [3].

If $p_n(t)=P(X_t=n)$, $n\in N$ represents the probability that at moment *t* the process is in the state *n*, then the probability that at some following point $t+\Delta t$ to be still in the state *n* will be:

$$p_n(t+\Delta t)=P(X(t+\Delta t)=n)=(1-\lambda\Delta t)p_n(t)+\lambda p_{n-1}(t)\Delta t+0(\Delta t), \quad (2)$$

The relationship (2) is equivalent to the following:

$$p_n(t+\Delta t)-p_n(t)=-\lambda\Delta t p_n(t)+\lambda p_{n-1}(t)\Delta t+0(\Delta t). \quad (3)$$

If the last relation is divided to Δt and then it proceeds to limit for $\Delta t\to 0$, we obtain the differential equations that characterize the Poisson process:

$$\frac{dp_n}{dt}=-\lambda p_n(t)+\lambda p_{n-1}(t),\ n\geq 1 \quad (4)$$

And for $n=0$, $p_{n-1}(t)=0$ and

$$\frac{dp_0}{dt}=-\lambda p_0(t) \quad (5)$$

The solutions of equations (4) and (5) are:

$$p_0(t)=e^{-\lambda t} \quad \text{and} \quad p_n(t)=\frac{(\lambda t)^n}{n!}e^{-\lambda t}, \quad (6)$$

and they represent the probability that at the time $t\geq 0$, the random variables X_t with the Poisson distribution to be in the state *n*.

The entire demonstration of the above relation is in [4], in the chapter dedicated to stochastic processes.

Observation:

In accordance with [3], these remarks regarding a Poisson process can be made:

* is a process whose streets are natural numbers,

* its increases are independent over an interval with length Δt,

* $\sum\limits_{n\in N^*}p_n(t)=1$ (at the time *t*, the process is certainly in one of the states $x\in N^*$);

* is a process of counting, its states are often identified by the number of occurrences of a particular event within an interval with length *t*.

*the average of the process is $E(X_t)=\lambda t$.

B)The Yulle-Furry process is a linear process of birth, with $X_t=n$ and the property that each person in the interval $(t,t+\Delta t)$

independently of the other can give birth to another person with the probability $\lambda\Delta t + 0(\Delta t)$.

Then the variables have binomial distribution and the probability that the *n* people give birth to other *k* people is

$$P(X_{t+\Delta t} = k) = p_k(t+\Delta t) = C_n^k\left(\lambda\Delta t + 0(\Delta t)\right)^k\left(1-\lambda\Delta t - 0(\Delta t)\right)^{n-k} \quad (7)$$

In particular, for $k=1$,

$$p_1(t) = \lambda n\Delta t + 0(\Delta t) \quad (8)$$

The differential equations of the process are:

$$\frac{dp_n}{dt} = -\lambda n p_n(t) + \lambda(n-1)p_{n-1}(t),\ n \geq 1 \quad (9)$$

$$\frac{dp_1}{dt} = -\lambda p_1(t) \quad (10)$$

With the solutions $p_1(t) = e^{-\lambda t}$ and

$$p_n(t) = e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1} \quad (11)$$

A characteristic of the birth (this observation is made in [5]) and death processes is that if E_n is the state of process in the interval $(0,t)$, then in $(t,t+\Delta t)$ the system can either be in the state E_{n+1} or in the state E_{n-1}.

In [4], Orman G.V. describes The Furry-Yulle process. This process can be characterized by the following conditions:

a)For $n \geq 1$, from the state E_n, the process can pass either in the state E_{n+1}, or in E_{n-1}; and from the state E_0 only in E_1.

b)If $X_t = E_n$ (at time *t* the process is in the state E_n), then the probability that in the interval $(t,t+\Delta t)$ to be in the state E_{n+1} is $\lambda_n\Delta t + 0(\Delta t)$;

c)The probability that in the interval $(t,t+\Delta t)$ to be in the state E_{n-1} is $\mu_n\Delta t + 0(\Delta t)$;

d)The probability that in the interval $(t,t+\Delta t)$ to have more changes than previous ones is $0(\Delta t)$, with $\lim\limits_{\Delta t\to 0}\frac{0(\Delta t)}{\Delta t} = 0$.

The differential equations that characterize the Poisson process are:

$$\frac{dp_n}{dt} = -(\lambda_n + \mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu_{n-1}p_{n+1}(t),\ n \geq 1 \quad (12)$$

$$\frac{dp_0}{dt} = -\lambda_0 p_0(t) + \mu_1 p_1(t) \quad (13)$$

One of the areas where properties of the Yulle-Furry processes can be applied is serving systems theory.

D.Kendall [2] defines such a system through three elements: 1) input stream, 2) order serving; 3) serving mechanism.

*In case of the input stream (M), the interval between two arrivals $t_n - t_{n-1}, n \geq 1$ is a random variable with the Poisson distribution, so the probability that in the interval $X_t = n$ is $p_n(t) = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$ and λ is the average number of requests in the time unit and the average time interval between two consecutive applications could be $\frac{1}{\lambda}$.

* Regarding the order of the servers it is assumed that it coincides with the order the arrivals;

*Regarding the mechanism of serving, it can be observed that the durations of serving are independent random variables that do not depend on the input stream, there were generally serving lines with the servings flow intensity μ and the average time between two servings being $\frac{1}{\mu}$;

* If both the order of serving and the serving mechanism are of poissonian type and there is only one service station, then the model type is M/M/1. It will be demonstrated by the application 2) that such a model is Yulle-Furry process.

It is considered the case of statistical equilibrium with $p_n(t) = p_n =$ constant (the probability of *n* server requests in system). Then the differential equations (12) and (13) become:

$$p_n'(t) = -(\lambda+\mu)p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t) \quad (14)$$

$$-\lambda p_0 + \mu p_1 = 0 \quad (15)$$

$$-(\lambda+\mu)p_n + \lambda p_{n-1} + \mu p_{n+1} = 0 \quad (16)$$

From the relation (15) it obtains:

$$p_1 = \frac{\lambda}{\mu} p_0 = \rho p_0 \ , \quad (17)$$

where $\rho = \frac{\lambda}{\mu}$ is the serving factor of a station and $\rho^* = \frac{\rho}{s}$ is the traffic intensity.

Using the recurrence (16) the solution is obtained by mathematical induction:

$p_n = \frac{1}{n!}\rho^n p_0, n \in N$, with

$$p_0 = \left(1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + ... + \frac{\rho^s}{s!}\right)^{-1} \quad (18)$$

If $\rho > 1$, then the average number of existing units in system at the time *t* in the case of *s* service stations is calculated using the next relationship:

$$T = \rho - \frac{\rho^{s+1}}{(s+1)!} p_0 \quad (19)$$

The average waiting time in line to start serving:

$$S = T\frac{1}{\lambda} \quad (20)$$

And the total average waiting time is:

$$S_t = S + \frac{1}{\lambda} \quad (21)$$

RESULTS AND DISCUSSIONS

1. It examines the efficiency of a vaccine used to prevent of animals from certain diseases and is believed that to organisms don't act systematic factors but only random ones. If the arrival at the vaccination center is in accordance with a Poisson process by parameter $\lambda = 70$ animals / week and the probability to be adverse reactions is 1/7, what is the probability that in 4 consecutive weeks not to happen any adverse effects at any vaccinated animals?

Solution: It is noted with X_t the volume of the vaccinated animals at time t, which can be grouped into ones who have shown adverse effects $\{X_1(t)\}_{t\geq 0}$ and those which are not shown $\{X_2(t)\}_{t\geq 0}$.

It is known that the average of process is $\lambda = 70$ and we notice that the defined process verifies the conditions a), b), c) specific to a Poisson process.

We remark that can be made some observations regarding the process X_t:

-It is noticed that of the states of the process are natural numbers (number of animals vaccinated),

-Its increases are independent over an interval with length Δt,

- $\sum_{n\in N^*} p_n(t) = 1$ (at the time *t*, the process is certainly in one of the states $x \in N^*$);

-It is a process of counting, its states are often identified by the number of occurrences of a particular event within an interval with length *t* (the number of vaccinated animals with adverse effects)

-The average of the process is $E(X_t) = \lambda t$.

From the statement it follows that the number of the vaccinated animals who showed adverse effects $\{X_1(t)\}_{t\geq 0}$ is a Poisson process with parameter $\lambda_1 = \frac{1}{7}\lambda = \frac{1}{7}\cdot 70 = 10$. Then, the probability within 4 weeks not to occur any adverse effects to a vaccinated animal (i.e. the process $X_1(t)$ with the Poisson distribution to be in the state $n = 0$) is obtained by applying formula $p_0(t) = e^{-\lambda_1 t}$. Then,

$p_0(4) = P(X_1(4) = 0) = e^{-4\lambda_1} = e^{-40}$.

This result can be interpreted that the probability for 4 weeks may not occur adverse effects in any of the vaccinated animals under the circumstances is $e^{-40} = 4.24e^{-18}$.

If we want to know the probability that a certain number of vaccinated animals to manifest adverse effects in an interval of length *t*, then the answer could be give applying the formula $p_n(t) = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$.

For example, the probability that in 2 consecutive weeks to happen adverse effects at one vaccinated animal is

$$p_1(2)=\frac{(2\lambda_1)^1}{1!}e^{-\lambda_1}$$

$$p_1(2)=\frac{(2\lambda_1)^1}{1!}e^{-2\lambda_1}=20e^{-20}.$$

2. At a warehouse for raw materials arrive an average of 80 requests per day and are served on average 50 applications per day. If the deposit has 2 serving stops, we have to determine:

a) The probability p_4 in order to be 4 applications in system;

b)The average number of requests from the system;

c) The average waiting time in line to start serving;

d) The average total waiting time.

Solution: The situation fits in a M/M/2 model, with $\lambda=80$ (there are in average of 80 requests per day) and $\mu=50$ (i.e. in average 50 requests per day are served). If Y_t is the number of served requests until the moment *t*, then: $P(Y_t=n)=\frac{(\mu t)^n}{n!}e^{-\mu t}$.

a)The report $\rho=\frac{\lambda}{\mu}=\frac{80}{50}=1.6>1$ and $\rho^*=\frac{\rho}{s}=0.8<1$, the traffic intensity.

Using (18) it results:

$$p_0=\left(1+\rho+\frac{\rho^2}{2}\right)^{-1}=0.2577,$$

$$p_4=\frac{\rho^4}{4!}p_0=7\%.$$

b) The average number of existing units in system at the time *t* is calculated with the relation (19)

$$T=\rho-\frac{\rho^{s+1}}{(s+1)!}p_0=\rho-\frac{\rho^3}{3!}p_0=1.4\text{ requests.}$$

c) Using (20) we can determine the average waiting time in line of a request

$$S=T\frac{1}{\lambda}=0.0175\text{ days}$$

With (21) we obtain the total average waiting time

$$S_t=S+\frac{1}{\mu}=0.0375\text{ days.}$$

CONCLUSIONS

The examples discussed in this paper lead us to conclude that the stochastic processes can be the optimal solution to solve specific problems of agriculture.

Thus, if it is analyzed the evolution of a phenomenon in relation to time and we find that the results have a Poissonian distribution, then we can calculate the probability that in a given time, the event to happen a certain number of times.

In other cases, if the situation of the agriculture can be transposed in to a serving model with known inputs, outputs and serving mechanism, then the parameters associated can be determined by applying certain properties of stochastic processes.

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